The Era of polygons

By Grandmaster Mark Aarøe Nissen

August 2022

World Pi Federation

www.worldpifederation.org

Introduction

The goal of this paper is to investigate how polygons were used to approximate the value of π through out the time of our history. Firstly, the main focus shows how to bisect polygons and the mathematics in which it's based on. The method of Archimedes will be explained and used. The works of the mathematicians after him such as Zu Chongzhi and Ludolf van Ceulen will be demonstrated as well. The rational approximations for π will also be demonstrated.

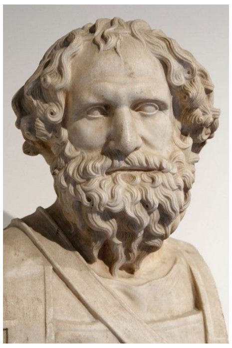
Finally, a description and a proof of Isaac Newton's discovery in 1666 will be explained and why it ended the *Era of polygons.* I will formulate two theorem: one of the *Archemedian iteration* and the second is *Newton's approximation.* The first one is on how to bisect polygons and the second one is on how to use infinitesimal calculus to get the value of π much faster. Furthermore, I will write a program in R that successfully calculate the digits as a result of these two theorems.

Early History

The number π has been known for thousands of years. The ancient Babylonians stated that $\pi \approx 3.12$ around 1900 BC and the old Egyptian text Rhind Papyrus, from about 1650 BC, states that $\pi \approx 3.16$. The first decimal digit of π were known.

The first theoretical calculation of π was done by Archimedes of Syracuse (287–212 BC). He approximated the perimeter of a circle with the perimeters of two regular polygons: the polygon inscribed within the circle and the polygon within which the circle was circumscribed. Since the actual perimeter of the circle lies between the perimeters of the inscribed and circumscribed polygons, the perimeters of the polygons gave upper and lower bounds for the perimeter of the circle. He used a $3 \cdot 2^5 = 96$ polygon to calculate the first 2 decimal digits of π. From this approach he concluded that $\frac{223}{71} < \pi < \frac{22}{7}$ and thus $\pi \approx \frac{22}{7}$ which is approximately equal to 3.142857.

Archimedes lived in the Hellenistic city of Syracuse which is located on the eastern coast of Sicily. During the Second Punic War the Roman Republic stormed the city after a protracted siege and Archimedes was killed by a Roman soldier. According to legend his last words were *"Do not disturb my circles"*.



Statue of Archimedes

His approach however didn't die with him. The *Era of Polygons* had begun: The approach of approximate the value of π by bisecting a circle with polygons. The method was used by π-digit hunters after him to obtain more accurate approximations. Claudius Ptolemy $(85 - 165)$ of Alexandria used a 360-polygon to get the approximation $\pi \approx 3\frac{17}{120}$ which is approximately equal to 3.14166 and thus the fist 3 decimal digits where known.

The Chinese mathematician Liu Hui $(225 - 295)$ independently developed a similar method to the one employed by Archimedes. He used a $3 \cdot 2^{10} = 3072$-sided polygon to calculate the first 5 decimal digits of π which is approximately equal to 3.14159 and a more accurate approximation than the one calculated by Archimedes or Ptolemy.

The Chinese mathematician Zu Chongzhi $(429 - 501)$ continued the work of Liu Hui. He used a $3 \cdot 2^{13} = 24576$ sided polygon to calculate the first 7 decimal digits of π which is 3.1415926. It was a record in accuracy which would not be surpassed for over 900 years. We also know that he gave a rational approximation of $\pi \approx \frac{355}{113}$, also known as the *Zu's ratio*, which is approximately equal to 3.14159292.

Zu Chongzhi wrote a mathematical paper entitled *Zhui Shu* (Methods for Interpolation). Sadly this paper has been lost since the Song Dynasty $(960 - 1279)$. It contained many of his mathematical achievements included his work on π. It also contained other interesting works. We know that he was calculating one year as 365.24281481 days, which is very close to 365.24219878 days as we know today. He was also calculating the Jupiter year as about 11.858 Earth years, which is very close to 11.862 as we know of today.

Painting of Zu Chongzhi

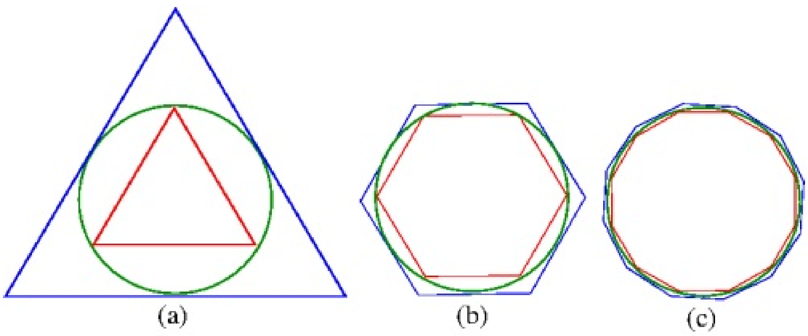
Further progress was not made for nearly a millennium, until the 15th century. In 1424 the Persian mathematician Jamshīd al-Kāshī $(1380-1429)$ used a $3 \cdot 2^{28}$ polygon to calculate the first 16 decimal digits of π which is 3.1415926535897932 which stood as the world record for about 180 years. This record would later be broken by the Dutch mathematician Ludolf van Ceulen $(1540-1610)$ who made a spectacular effort in bisecting polygons. We will get back to him later.

Bisecting polygons

Archimedes' method for finding an approximate value for π is based on bisecting polygons. We can draw a polygon with k sides that circumscribes the circle. Let the perimeter of such a polygon be denoted by C_k. We can also draw a polygon with k sides that inscribe the same circle. Let the perimeter of such a polygon be denoted by c_k. Then, if C denotes the circumference of the circle we have that

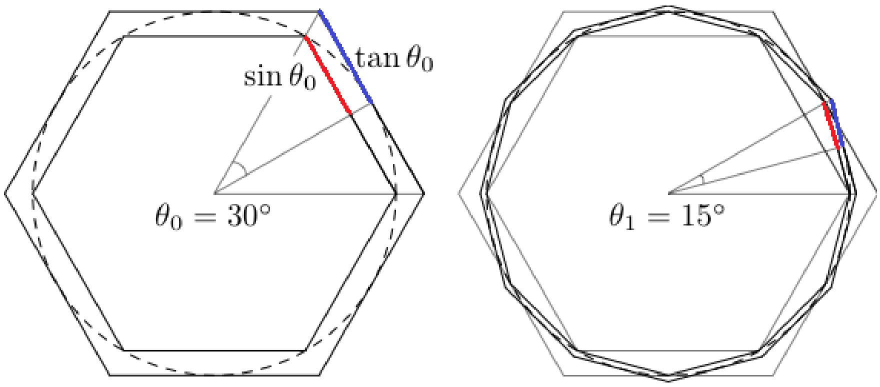
$$c_k < C < C_k.$$

As we increase the value for k both c_k and C_k get closer to C. This is illustrated on the picture below for the cases with $k = 3, 6, 12$. You can make your own polygon on *www.worldpifederation.org/polygon*.



We see the inscribed polygon (red) and the circumscribed polygon (blue) around the circle (green).

The perimeter of the circle with diameter 1 equals the value of π, since the perimeter of a circle equals its diameter times π by definition. Consider the case of a circle with radius 1. We see that each side of a regular inscribed hexagon ($k = 6$) has length 1 and each half-side has length 1/2. Let $\theta_0 = 30°$ be our start-angle. This is reflected in the fact that $\sin(30°) = 1/2$ (red line) which is proven by the figure below.



Consider a unit triangle with length 1, angles $60°$. Taking half of it, then it has $30°$. From Pythagoras' Theorem we have that $\cos(\theta_0) = \sqrt{1 - \sin(\theta_0)^2}$. We obtain that $\cos(30°) = \sqrt{1 - \frac{1}{2}} = \sqrt{3}/2$ and $\tan(30°) = \frac{\sin(30°)}{\cos(30°)} = \frac{1/2}{\sqrt{3}/2} = \sqrt{3}/3$ (blue line).

Let a_1 be the semi-perimeter of the regular circumscribed hexagon of a circle with radius one, and let b_1 denote the semi-perimeter of the regular inscribed hexagon. By examining the figure we see that we have exactly 6 of these right triangles and thus $a_1 = 6\tan(30°) = 2\sqrt{3} = 3.464101...$ and similarly $b_1 = 6\sin(30°) = 3$. Before we continue let's consider the trigonometric identities.

The trigonometric identities

For an angle $\alpha \notin \{0°, 90°, 180°, 270°, 360°\}$ we have that

1. $\sin(\alpha/2) = \sqrt{\frac{1-\cos(\alpha)}{2}}$ and $\sin(2\alpha) = 2\cos(\alpha)\sin(\alpha)$.
2. $\cos(\alpha/2) = \sqrt{\frac{1+\cos(\alpha)}{2}}$.
3. $\tan(\alpha/2) = \frac{\sin(\alpha)}{1+\cos(\alpha)} = \frac{\tan(\alpha)\sin(\alpha)}{\tan(\alpha)+\sin(\alpha)}$.

Now consider a 12-sided regular circumscribed polygon of a circle with radius 1 and a 12-sided regular inscribed polygon. Their semi-perimeter will be denoted a_2 and b_2 respectively. The angles are halved, but the number of sides is doubled. Thus $a_2 = 12\tan(15)°$ and $b_2 = 12\sin(15°)$.

Since $\cos(15°) = \sqrt{\frac{1+\cos(30°)}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}}$, we have the following

$$\sin(15°) = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$b_{12} = 12 \cdot \sin(15°) = 12\left[\frac{\sqrt{2-\sqrt{3}}}{2}\right] \approx 3.1058285.$$

$$\tan(15°) = \frac{\sin(30°)}{1+\cos(30°)} = \frac{1/2}{1+\sqrt{3}/2} = \frac{1}{2+\sqrt{3}},$$

$$a_{12} = 12 \cdot \tan(15°) = 12 \cdot \left[\frac{1}{2+\sqrt{3}}\right] \approx 3.2153903.$$

Archimedes calculate up until $k = 96$. A 96-polygon. His result was

$$b_{96} = 96\left[\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2}\right] \approx 3.14103195$$

and

$$a_{96} = 96\left[\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}\right] \approx 3.14271460.$$

Taking square roots of square roots is extremely time consuming. This is were Archimedes stopped.

Let's generals it all. In general, after k steps of doubling, denote the semi-perimeters of the regular circumscribed and inscribed polygons for a circle of radius 1 with $3 \cdot 2^k$ sides as a_k and b_k respectively and let $\theta_k = 60°/2^k$. Then

$$a_k = 3 \cdot 2^k \tan(\theta_k) \quad \text{and} \quad b_k = 3 \cdot 2^k \sin(\theta_k).$$

Lets employ the following iteration, which permits these values to be calculated by simple formulas involving only arithmetic and square roots.

Theorem 1. *The Archimedean iteration for π.* Define the sequences of real numbers A_k, B_k by the following: $A_1 = 2\sqrt{3}, B_1 = 3$. Then, for $k \geq 1$, set

$$A_{k+1} = \frac{2A_k B_k}{A_k + B_k}, \quad B_{k+1} = \sqrt{A_{k+1} B_k}.$$

Then for all $k \geq 1$, we have $A_k = a_k$ and $B_k = b_k$.

Proof. Note that $A_1 = a_1$ and $B_1 = b_1$. By induction we assume that the result is true for some k. Using the *trigonometric identities* we get

$$\begin{aligned} A_{k+1} &= \frac{2A_k B_k}{A_k + B_k} = \frac{2 \cdot 3 \cdot 2^k \tan(\theta_k) \cdot 3 \cdot 2^k \sin(\theta_k)}{3 \cdot 2^k \tan(\theta_k) + 3 \cdot 2^k \sin(\theta_k)} \\ &= 3 \cdot 2^{k+1} \cdot \frac{\tan(\theta_k)\sin(\theta_k)}{\tan(\theta_k) + \sin(\theta_k)} \underset{\substack{\uparrow \\ \text{trigonometric identity 3}}}{=} 3 \cdot 2^{k+1} \tan(\theta_k/2) \\ &= 3 \cdot 2^{k+1} \tan(\theta_{k+1}) = a_{k+1}. \end{aligned}$$

Similarly we get that

$$\begin{aligned} B_{k+1} &= \sqrt{A_{k+1} B_k} = \sqrt{9 \cdot 2^{2k+1} \tan(\theta_{k+1}) \sin\theta_k} \\ &\underset{\substack{\uparrow \\ \text{trigonometric identity 1}}}{=} \sqrt{9 \cdot 2^{2k+2} \tan(\theta_{k+1}) \sin(\theta_{k+1}) \cos(\theta_{k+1})} \\ &= \sqrt{9 \cdot 2^{2k+2} \sin^2(\theta_{k+1})} \\ &= 3 \cdot 2^{2k+1} \sin(\theta_{k+1}) = b_{k+1}. \end{aligned}$$

Which complete the induction steps and finish the proof. $\square$

The R Code

A code has been written in the program R to compute the first 25 iterations of Archimedes algorithm.

```
# Number of iterations
n=25

# Save the values for a and b
a=c()
b=c()
a[1]=2*sqrt(3)
b[1]=3
for(i in 2:n)   {
        a[i]=2*a[i-1]*b[i-1]/(a[i-1]+b[i-1])
        b[i]=sqrt(a[i]*b[i-1])
}

# Print the result for both a and b with 20
   digits each.
cbind(sprintf("%.20f",b),sprintf("%.20f",a))
```

For each iteration k we calculated the values for a_k and b_k. The digits wich are the same are hightlighted with the orange colour. The semi-perimiter is given for both a_k and b_k.

The R output

	Sides	**Inscribed polygon**	**Circumscribed polygon**
k	$3 \cdot 2^k$	b_k	a_k
1	6	**3**.00000000000000000000	**3**.46410161513775438635
2	12	**3**.10582854123024931781	**3**.21539030917347279370
3	24	**3.1**3262861328123820570	**3.1**5965994209750045130
4	48	**3.1**3935020304686718262	**3.1**4608621513143482673
5	96	**3.14**103195089050934996	**3.14**271459964536825638
6	192	**3.141**45247228546198315	**3.141**87304997982419508
7	384	**3.141**55760791185745262	**3.141**66274705684855917
8	768	**3.141**58389214831812453	**3.141**61017660468955270
9	1536	**3.14159**046322805002305	**3.14159**703432152603853
10	3072	**3.14159**210599927130048	**3.14159**374877135144644
11	6144	**3.141592**51669215722202	**3.141592**92738509643428
12	12288	**3.141592**61936538358739	**3.141592**72203861350548
13	24576	**3.1415926**4503369062282	**3.1415926**7070199765826
14	49152	**3.14159265**145076727066	**3.14159265**786784436258
15	98304	**3.14159265**305503643262	**3.14159265**465930559458
16	196608	**3.141592653**45610361209	**3.141592653**85717079155
17	393216	**3.141592653**55637007389	**3.141592653**65663653569
18	786432	**3.141592653**58143668934	**3.141592653**60650330479
19	1572864	**3.1415926535**8770323218	**3.1415926535**9396977502
20	3145728	**3.1415926535**8926997891	**3.1415926535**9083672564
21	6291456	**3.1415926535**8966122150	**3.1415926535**9005290819
22	12582912	**3.141592653589**75936522	**3.141592653589**85706484
23	25165824	**3.141592653589**78379012	**3.141592653589**80821503
24	50331648	**3.14159265358979**000737	**3.14159265358979**622462
25	100663296	**3.14159265358979**133964	**3.14159265358979**311600

Rational approximations

We know that Archimedes claimed that $\frac{223}{71} < \pi < \frac{22}{7}$ and Zu Chongzhi claimed that $\pi \approx \frac{355}{113}$. The true decimal representation of approximations are $\pi = 3.1415926535...$, $\frac{22}{7} \approx 3.142857$, $\frac{223}{71} \approx 3.140845$ and $\frac{355}{113} \approx 3.14159292$.

We want to find a rational number that is very close to our polygon-values. Consider the value $b_{96} \approx 3.14103195$. A rational number approximating this number is $314,103/100,000$. One approach to find a good rational approximation for any number is to write it in the form of a continued fraction expansion which we can truncate after some term. A continued fraction exspansion is given by the following form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots}}}}$$

where $a_0, a_1, ...$ are natural numbers. For an irrational number, the series goes on for ever while for a rational number it would end after a finite number of terms. There is a simple procedure to write down it's continued fraction expansion. For illustration let us consider the number $3.14 = 157/50$. We see that $a_0 = 3$. The remaining part of the exspansion has value $7/50$. To find a_1 we have to take the integer part of $50/7$. This gives us $a_1 = 7$. Repeating the procedure with remainder, that is $1/7$, we get $a_2 = 7$ and thus we have that

$$3.14 = 3 + \frac{1}{7 + \frac{1}{7}}.$$

If we perform the procedure for $b_{96} \approx 3.14103195$, we get explicitly

$$\begin{aligned}
\frac{314,103,195}{10,000,000} &= 3 + \frac{14,103,195}{100,000,000} = 3 + \frac{1}{100,000,000/14,103,195} \\
&= 3 + \frac{1}{7 + 1,277,635/14,103,195} = 3 + \cfrac{1}{7 + \frac{1}{14,103,195/1,277,635}} \\
&= 3 + \cfrac{1}{7 + \frac{1}{11 + 49,210/1,277,635}} = 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{1,277,635/49,210}}} \\
&= 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{25 + 47,385/49,210}}} = 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{25 + \frac{1}{49,210/47,385}}}} \\
&= 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{25 + \frac{1}{1 + 1,825/47,385}}}} = 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{25 + \frac{1}{1 + \frac{1}{47,385/1,825}}}}} \\
&= 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{25 + \frac{1}{1 + \frac{1}{25 + 1760/1825}}}}} = 3 + \cfrac{1}{7 + \frac{1}{11 + \frac{1}{25 + \frac{1}{1 + \frac{1}{25 + \frac{1}{1825/1760}}}}}}
\end{aligned}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{11 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + 65/1760}}}}}} = 3 + \cfrac{1}{7 + \cfrac{1}{11 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{1760/65}}}}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{11 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{27 + 5/65}}}}}}} = 3 + \cfrac{1}{7 + \cfrac{1}{11 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{27 + \cfrac{1}{65/5}}}}}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{11 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{27 + \cfrac{1}{13}}}}}}}}.$$

By truncating the above series at various orders we get the following rational approximations $(r_1, r_2, r_3, ...)$ of b_{96}: $r_1 = 3 + \frac{1}{7+0} = \frac{22}{7}$, $r_2 = 3 + \frac{1}{7+\frac{1}{11+0}} = 3 + \frac{11}{78} = \frac{245}{78}$, $r_3 = 3 + \frac{1}{7+\frac{1}{11+\frac{1}{25+0}}} = 3 + \frac{1}{7+\frac{25}{276}} = 3 + \frac{276}{1957} = \frac{6147}{1957}$ and so on.

This does not include Archimedes' value 223/71. We don't know how he came up with this guess. As matter of fact $245/78 \approx 3.141025641$ is a better approximation to b_{96} rather than 223/71. Moreover we see that $245/78 < b_{96}$ and so we can use it as a lower bound. Applying the same procedure to a_{96} we get the rational approximations: $3/1, 22/7, 3149/1002$ and so on. In this case we see that $22/7 > a_{96}$ and so we can use this value as an upper bound. Hence we finally get that $245/78 < \pi < 22/7$.

Finally, we have found the continued fraction series for $a_{24576} \approx 3.14159267$ and we get the rational approximations: 3/1, 22/7, 333/106, 355/113, ... and so on. The last number is precisely the approximation discovered by Zu Chongzhi.

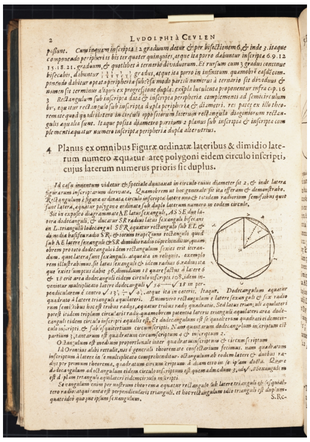
Ludolf van Ceulen - the great polygon creator

Ludolf van Ceulen (1540-1610) was born in Hildesheim, Germany. He emigrated to the Netherlands and moved to Delft around 1576. In 1580 he submitted a request to the Delft town council to be allowed to open a fencing school in the town. The Council agreed to his request

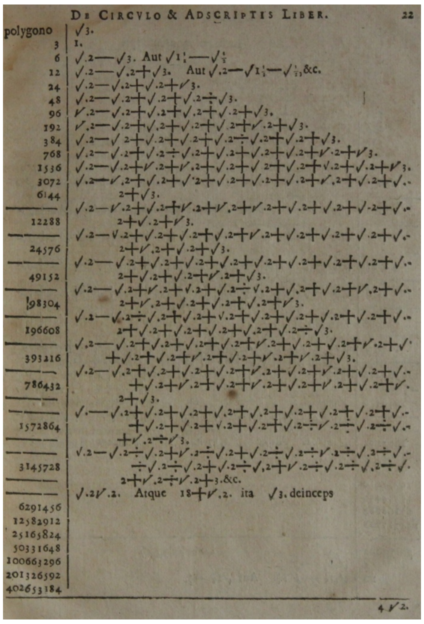
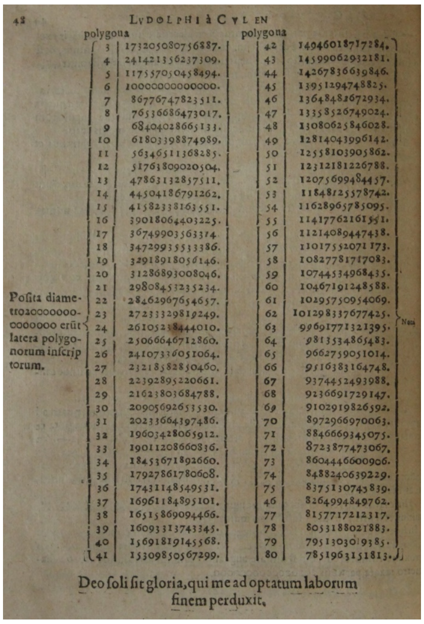
During his time in Delft, Van Ceulen was teaching mathematics and was involved in a number of mathematical disputes. Up to this time Van Ceulen had not read Archimedes' work on in which he had used a regular polygon of 96 sides to approximate the value of π. Van Ceulen had a problem since he could not read Greek nor Latin, but Jan Cornets de Groot, the burgomaster of Delft and father of the jurist, scholar, statesman and diplomat, Hugo Grotius, translated Archimedes' approximation to π for Van Ceulen around 1586. This proved a significant point in Van Ceulen's life for he spent the rest of his life obtaining better approximations to π using Archimedes' method with regular polygons with many sides.

Ludolf van Ceulen

In 1594 he moved with his wife and children to Leiden where again he taught mathematics and fencing. The same year he made a request of the Leiden Council that he be given permission to open a fencing school which the Council accepted. Nobody else was allowed to run a fencing school in Leiden and in 1602, when he realized that his assistant Pieter Bailly was running his own school, Van Ceulen complained to the Council who forced the closure of Bailly's fencing school. In addition to teaching mathematics and fencing, Van Ceulen was writing his most famous work, the book *Vanden circkel* (On the Circle) which he published in 1596 in Dutch. In this book he gave π correct to 20 decimal places using a regular polygon of $15 \cdot 2^{31}$ which was a world record at the time. Below are some pictures from his book.



In the table below, Van Ceulen lists some lengths of interest again using square roots. Specifically, he starts with an equilateral triangle which has three sides and each time calculates the length of a side of a regular polygon with double the number of sides as the previous one. The next table shows a similar calculation in which Van Ceulen has used a circle with very large diameter (200000000000000) and given the length of a side of each inscribed regular polygon. For example, an inscribed hexagon (6 sides) has side length exactly half the diameter of the circle.



42 LVDOLPHI à CVLEN

Posita diametro20000000000000000 erũt latera polygonorum inſcriptorum.

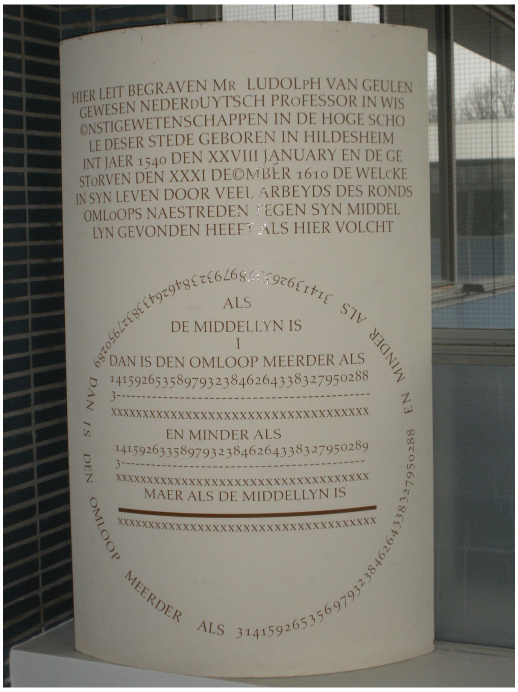
polygona		polygona	
3	17320508075 6887.	42	14946018717284.
4	24142135623 7309.	43	14599062932181.
5	11755705045 8494.	44	14267836639846.
6	100000000000000.	45	13951294748825.
7	86776747823511.	46	13648482672934.
8	76536686473017.	47	13358526749024.
9	68404028665133.	48	13080625846028.
10	61803398874989.	49	12814043996142.
11	56346511368285.	50	12558103905862.
12	51763809020504.	51	12312181226788.
13	47863132857511.	52	12075699484457.
14	44504186791262.	53	11848125578742.
15	41582338163551.	54	11628965785095.
16	39018064403225.	55	11417762161551.
17	36749903563374	56	11214089447438.
18	34729935533386.	57	11017552071173.
19	32918918056146.	58	10827781717083.
20	31286893008046.	59	10744534968435.
21	29808453235234.	60	10467191248588.
22	28462967654657.	61	10295750954069.
23	27233329819249.	62	10129833767742 5.
24	26105238444010.	63	99691771321395.
25	25066646712860.	64	98135348654483.
26	24107336051064.	65	96627590510 14.
27	23218582850460.	66	95163831647 48.
28	22392895220661.	67	93744524939 88.
29	21623803684788.	68	92366917291 47.
30	20905692653530.	69	91029198265 92.
31	20233664397486.	70	89729669700 63.
32	19603428065912.	71	88466693450 75.
33	19011208660836.	72	87238774730 67.
34	18453671892660.	73	86044466009 06.
35	17927861780608.	74	84882406392 29.
36	17431148549531.	75	83751307438 39.
37	16961184895101.	46	82649948497 62.
38	16515869094466.	77	81577172123 17.
39	16093313743345.	78	80531880218 83.
40	15691819145568.	79	79513030938 5.
41	15309850567299.	80	78519631518 13.

Nota

Deo ſoli ſit gloria, qui me ad optatum laborum finem perduxit.

In 1600 he was appointed the first professor of mathematics at the Engineering School. The School was set up in the Faliedenbegijnkerk so Van Ceulen could teach fencing and mathematics in the same building. For the last ten years of his life he taught arithmetic, surveying and fortification in the Engineering School and he continued to compute more digits of π. After almost 25 years of work he had computed π to 35 places which he did using a polygon with 2^{62} sides, that is 4,611,686,018,427,387,904 sides. It was a new world record.

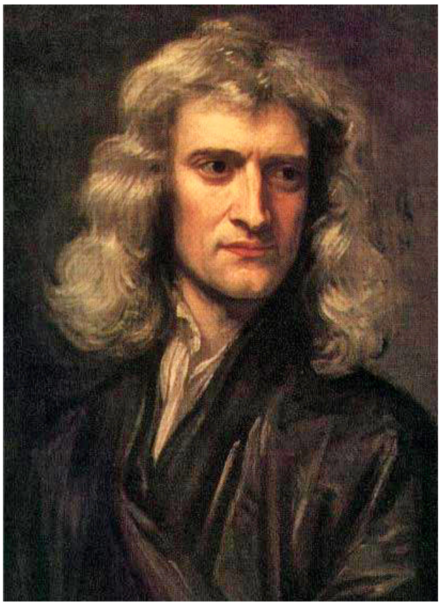
Ludolf van Ceulen died in Leiden in 1610. He was buried in St Peter's Church in Leiden and his approximations for π were engraved on his original tombstone which went missing around the year 1800. Since 2000 a modern version stands in St Peter's Church and carved on its tombstone is his lower bound and upper bound of π.



Van Ceulen is famed for his calculation of π to 35 places. Having published 20 places of π in his book *Vanden circkel* of 1596, the more accurate results were only published after his death. Van Ceulen's work was translated into Latin after his death by his student Willebrord Snellius (1580-1626) and published in 1619 with the title *De circulo et adscriptis liber*. The translation into latin made his work more accessible to the mathematicians around the world. He was surpassed by the Austrian astronomer and mathematician Christoph Grienberger (1561-1636) who calculated the first 38 decimal digits of π. He was the last to use polygons to approximate the value of π and nobody was bisecting polygons ever again.

Newton's discovery

Isaac Newton (1642-1727) was a British mathematician, physicist, astronomer and much more. The year was 1666 and Newton was just 23 years old. He was quarantining at home due to an outbreak of bubonic plague. He had just developed what he called the *Theory of Fluxions* what we today know as infinitesimal calculus and he was playing around with the binomial theorem which were known at his time.



Painting of Isaac Newton

Binomial Theorem

Let $\alpha \in \mathbb{R}$ and let $x \in \mathbb{R}$ such that $|x| < 1$. Then

$$\begin{aligned}(1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{k=0}^{n-1} (\alpha - k) \right) x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + ...\end{aligned}$$

where $\binom{\alpha}{n}$ denotes a binomial coefficient.

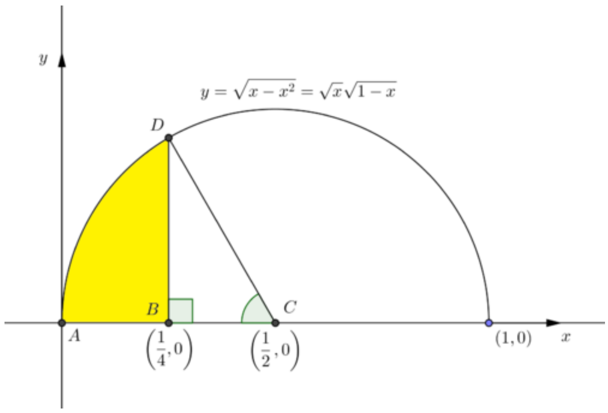
Newton saw by letting $n = 1/2$ we would get $\sqrt{3} = \sqrt{4-1} = \sqrt{4(1-\frac{1}{4})} = 2\sqrt{1-\frac{1}{4}} = 2(1-\frac{1}{4})^{1/2} = 2\left[1 + \frac{1}{2}(-\frac{1}{4}) + \frac{1}{2}(-\frac{1}{2})(-\frac{1}{4})/2! + ...\right]$ letting him be able to calculate the square root of 3 very fast because he get a very rapidly converging series expansion that will quickly give him the square root of 3 to high accuracy. Newton was particularly interested in the case for $n = 1/2$ because the equation for a unit circle is $x^2 + y^2 = 1 \Rightarrow y = (1 - x^2)^{1/2}$. The top part of the circle. This is basically the same expression he's been looking at. So now he had two different ways of representing the same ting and this motivated him to work further and he came up with a fruitful formula for calculating π.

Theorem 2. *Newton's approximation for π.*

$$\pi = \frac{3\sqrt{3}}{4} - 24\left(\sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+2}(n!)^2(2n-1)(2n+3)}\right)$$

Proof. We will break the proof up in two parts. The first part we will calculate the yellow area using calculus and in the second part we will calculate the same area using plane geometry. Finally we will combine the two expressions with each other.

Part 1. Let $\mathcal{A}$ denote the area of the yellow shaded region in the figure below.



Consider the semicircle embedded in the plane with radius $\frac{1}{2}$ and whose center is the point $\left(\frac{1}{2}, 0\right)$. From the equation of a circle we have:

$$\left(x - \frac{1}{2}\right)^2 + (y - 0)^2 = \left(\frac{1}{2}\right)^2.$$

Thus

$$y = \sqrt{(x - x^2)} = \sqrt{x}\sqrt{1-x} = \sqrt{x}\,(1-x)^{1/2}.$$

We will look at the last term and simplify it. From the binomial theorem we have that

$$
\begin{aligned}
(1-x)^{1/2} &= \sum_{k=0}^{\infty} \frac{1/2\cdot(1/2-1)\cdot(1/2-2)\cdot(1/2-3)\cdot\cdots\cdot(1/2-(k-1))}{k!}(-1)^k x^k \\
&= \sum_{k=0}^{\infty} \frac{1\cdot(1-2)\cdot(1-2\cdot 2)\cdot(1-2\cdot 3)\cdot\cdots\cdot(1-2(k-1))}{2^k k!}(-1)^k x^k \\
&= \sum_{k=0}^{\infty} \frac{1\cdot(-1)\cdot(-3)\cdot(-5)\cdot\cdots\cdot(-1)\cdot(2(k-1)-1)}{2^k k!}(-1)^k x^k.
\end{aligned}
$$

If k is even the fraction is negative and $(-1)^k$ is positive, thus the product is negative. If k is odd, then the fraction is positive and $(-1)^k$ is negative and thus the product is negative. Since the product then always will be negative we can replace the term $(-1)^k$ with (-1).

We also see that it's a product of odd numbers. For a natural number n we can write the product for odd numbers as

$$
1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1) = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot(2n-1)(2n)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \frac{(2n)!}{2^n(1\cdot 2\cdot 3\cdot\cdots\cdot n)} = \frac{(2n)!}{2^n n!}.
$$

In our case we have $n = k-1$, since our last term is $2(k-1)-1$. Therefore we have

$$
\begin{aligned}
(1-x)^{1/2} &= -\sum_{k=0}^{\infty} \frac{\frac{(2k-2)!}{2^{k-1}(k-1)!}}{2^k k!} x^k \\
&= -\sum_{k=0}^{\infty} \frac{(2k)!/[(2k-1)(2k)]}{2^{k-1}(k-1)!2^k k!} x^k \\
&= -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)(2k)2^{k-1}(k-1)!2^k k!} x^k \\
&= -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)2^{2k}(k!)^2} x^k.
\end{aligned}
$$

Then

$$
\sqrt{x}\sqrt{1-x} = -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)2^{2k}(k!)^2} x^{k+\frac{1}{2}}.
$$

We can calculate the areal of the area $\mathcal{A}$ by calculating it as the definite integral between $x = 0$ and $x = \frac{1}{4}$:

$$\begin{aligned}
\mathcal{A} &= \int_0^{\frac{1}{4}} \sqrt{x}\sqrt{1-x}dx \\
&= \left[-\sum_{k=0}^{\infty} \frac{(2k)!}{(k+3/2)(2k-1)2^{2k}(k!)^2} x^{k+\frac{3}{2}} \right]_0^{\frac{1}{4}} \\
&= -\sum_{k=0}^{\infty} \frac{(2k)!}{(k+3/2)(2k-1)2^{2k}(k!)^2 \cdot 2^{k+3/2} \cdot 2^{k+3/2}} \\
&= -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+3)(2k-1)2^{4k+2}(k!)^2}.
\end{aligned}$$

Part 2. Now we calculate $\mathcal{A}$ using the techniques of plane geometry. From the construction, we have that: $AC = CD$, $AB = BC$ and BD is common, so by the triangle Side-Side-side Equality: $\triangle ABD = \triangle CBD$ and thus $AD = AC = CD$ and so ACD is equilateral.

Thus we have that $\triangle BCD$ has angles $30°$, $60°$, $90°$. Hence by Pythagoras's Theorem: $BD^2 + BC^2 = CD^2$, we get

$$BD = \sqrt{CD^2 - BC^2} = \sqrt{(1/2)^2 - (1/4)^2} = \frac{1}{4}\sqrt{4-1} = \frac{\sqrt{3}}{4}.$$

Then we observe that $\mathcal{A}$ is the area $\mathcal{A}_S$ of the sector of the semicircle whose central angle is $60°$ subtracted the area $\mathcal{A}_T$ of the right triangle $\triangle BCD$.

$\mathcal{A}_S$ is $1/6$ of the area of the circle whose radius is $1/2$. Since the area of a circle is $A = r^2 \cdot \pi$ we have that

$$\mathcal{A}_S = \frac{1}{6}\left(\pi \cdot \left(\frac{1}{2}\right)^2\right)$$

And the area of the triangle is $\mathcal{A}_T = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{4}\right)$. Hence

$$\mathcal{A} = \mathcal{A}_S - \mathcal{A}_T = \frac{1}{6}\left(\pi \cdot \left(\frac{1}{2}\right)^2\right) - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{4}\right) = \frac{\pi}{24} - \frac{\sqrt{3}}{32}.$$

Thus

$$\frac{\pi}{24} - \frac{\sqrt{3}}{32} = -\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+3)(2k-1)2^{4k+2}(k!)^2}$$

and thus

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(-\sum_{k=0}^{\infty} \frac{(2k)!}{(2k+3)(2k-1)2^{4k+2}(k!)^2}\right)$$

which complete the proof. □

Newton used it to find π to 16 decimal digits by using only 22 terms of his formula. He then stopped. A code has been written in the program R to compute the first 22 iterations of Newton's algorithm. The same amount as he used. From the R output we see that Newton's approximated value is always higher than the true value of π, since last terms are subtracting. Similar we see that the sequence have a nearly linear convergence that can be noticed just by looking at them. For more terms we get 35 decimal digits just after 50 iterations and 66 correct digits after 100 terms.

The R Code for Newton's approach

```
# Number of iterations
n=22

k=0:n
pi_approx=(3*sqrt(3)/4) + 24 *(-1)*sum( factorial(2*k)
   /( (2*k+3)*(2*k-1)*2^(4*k+2)*(factorial(k))^2  ))

# Print the result with 20 digits
sprintf("%.20f", pi_approx )
```

n	Newton's approximation of π
1	3.29903810567665800590
2	3.14903810567665765063
3	3.14234167710522926953
4	3.14169063543856275089
5	3.14160740568004026585
6	3.14159508127348940931
7	3.14159307855742486737
8	3.14159273144802320132
9	3.14159266836317296878
10	3.14159265647217944561
11	3.14159265416506805479
12	3.14159265370679197105
13	3.14159265361396977667
14	3.14159265359485750935
15	3.14159265359086647962
16	3.14159265359002315421
17	3.14159265358984285399
18	3.14159265358980377414
19	3.14159265358979578053
20	3.14159265358979356009
21	3.14159265358979311600
22	3.14159265358979328618

Aftermatch

To match the computational power of Van Ceulen 2^{62}-sided polygon you would only need to compute the first 50 terms in Newton's method. This was the end of the *Era of polygons*. What before took years would now only take days. So no one was bisecting polygons to find π ever again.

In the time after Newton's discovery and the discovery of infinitesimal calculus many series were produced which converted to π or to a simple multiple of π. The most important one was derived by the British mathematician John Machin (1686-1751). He developed a fast approximation for π namely $\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$. He used this method to calculate the first 100 decimal digits of π in 1706 which was a world record. Later π-digit hunters used his method to find even more digits of π. More details about his approach in *https://worldpifederation.org/Fed/euler.pdf*.

The Danish mathematician Thomas Clausen (1801-1885) was born near Sønderborg, Denmark in 1801. He came from a poor family and could not read nor write at the age of 12. He got a job as caretaker at the local priest. They both had a passion for mathematics and astronomy and started working together. The work with the priest made Clausen more educated. At the age of 18 Clausen left Denmark and went to Hamborg where he worked as an assistent at Schumacher's observatorium. In 1827 he went to Münich to work at the Optical Institute as a calculator. In 1842 Clausen was hired by the staff of the Tartu Observatory, becoming its director in 1866-1872 and in 1869 he became an honorary doctor at the University of St. Petersburg. In 1885 he died in Tartu in modern day Estonia. The work by Clausen included studies of Number Theory, Astronomy and telegraphy. In 1847 he had calculated the first 248 decimal digits of π which was a world record at the time. Six years later in 1853 William Shanks (1812-1882) calculated the first 527 correct decimal digits of π correctly which was the world record until 1946.

With the development of computer technology in the 1950s, π was computed to thousands and then millions of digits. These computations were facilitated by the discovery of some advanced algorithms for performing the required high-precision arithmetic operations on a computer. For example, in 1965 it was found that the newly discovered fast Fourier transform could be used to perform high-precision multiplications much more rapidly. These methods dramatically lowered the computer time required for computing π to high precision. In spite of these advances, until the 1970s all computer evaluations of π still employed classical formulas, usually a variation of Machin's formula, however, in 1976 the *Brent-Salamin algorithm* was developed.

Today the *Chudnovsky algorithm* is by far the fasest algorithms used, as of the turn of the millennium, to calculate the digest of π. It was published by David Volfovich Chudnovsky (1947-present) and Gregory Volfovich Chudnovsky (1952-present), also known as the *Chudnovsky brothers*, in 1988. The algorithm is based on the negated Heegner number $d = -163$, the j-function $j(\frac{1+i\sqrt{163}}{2}) = -640320^3$, and on the following rapidly convergent generalized hypergeometric series:

$$\frac{1}{\pi} = 12\sum_{q=0}^{\infty} \frac{(-1)^q(6q)!(545140134q + 13591409)}{(3q)!(q!)^3(640320)^{3q+\frac{3}{2}}}.$$

The mathematics behind it is based on the work of the Indian mathematician Srinivasa Ramanujan $(1887 - 1920)$.

The Chudnovsky algorithm was used in the world record calculations of 2.7 trillion digits of π in December 2009, 10 trillion digits in October 2011, 22.4 trillion digits in November 2016, 31.4 trillion digits in September 2018–January 2019, 50 trillion digits on January 29, 2020, and 62.8 trillion digits on August 14, 2021. Finally the Japanese scientist Emma Haruka Iwao used it to calculate the first 100 trillion digits of π on March 21, 2022.

Why compute digits of π?

Certainly there is no need for computing π to millions or billions of digits in practical scientific or engineering work. We only need the first 6 decimal places of π to accurately calculate the circumference of the Earth to an error less than a meter and we only need the first 40 digits to calculate the circumference of the Milky Way galaxy to an error less than the size of a proton.

Back in the days computing digits of π was purely to show of mathematical strength. Today, however, one motivation for computing digits of π is that these calculations are excellent tests of the integrity of computer hardware and software. This is because if even a single error occurs during a computation, almost certainly the final result will be in error. On the other hand, if two independent computations of digits of π agree, then most likely both computers performed billions or even trillions of operations flawlessly.

The challenge of computing π has also stimulated research into advanced computational techniques. For example, some new techniques for efficiently computing linear convolutions and fast Fourier transforms, which have applications in many areas of science and engineering, had their origins in efforts to accelerate computations of π.

Another motivation for computing digits of π have long been of interest to mathematicians, who have still not been able to prove whether π is a *normal* number. If that was the case it would mean that no digit (or sequence of digits) occurs more frequently than any other. So any number you can possible imagine will appear in π somewhere.

Finally, there is a more fundamental motivation for computing π: *it is there.* The most famous and important constants of mathematics. Thus, as long as there are humans we will doubtless have more impressive computations of π.

Conclusion

In this paper we have investigated how polygons were used to approximate the value of π and demonstrated the mathematics in which it's all based on. We calculated the values of the inscribed and circumscribed polygons the cases $3 \cdot 2^1, .., 3 \cdot 2^{25}$ using the *Archemedian iteration* which was programmed in R. We also found the rational approximations of π. For a $3 \cdot 2^{13} = 24576$ we got $\frac{113}{355}$ which is the same result discovered by Zu Chongzhi. For a $3 \cdot 2^5 = 96$ polygon we got that $\frac{245}{78} < \pi < \frac{22}{7}$ which is a better approximation compared to Archimedes result which was $\frac{223}{71} < \pi < \frac{22}{7}$. A brief history of Ludolf van Ceulen's work was demonstrated as well.

Finally, a description and a proof of Isaac Newton's discovery in 1666 was completed and the results was programmed in R. The speed of compute digits after Newton's discovery was much faster and this explains why it ended the *Era of polygons*.

References

1: Azarian, Mohammad K. (2010). "al-Risāla al-muhītīyya: A Summary". Missouri Journal of Mathematical Sciences.

2: Needham, Joseph. Science and Civilisation in China, Volume 3, Mathematics and the Sciences of the Heavens and the Earth. Cambridge University

3: https://mathshistory.st-andrews.ac.uk/Biographies/Ptolemy/

4: Picture of Van Ceulen https://mathematics.edwardworthlibrary.ie/conic-sections/van-ceulen/

5: Picture of Archimedes and work: https://myhero.com/archimedes-3 and

6: https://thestrategybridge.org/the-bridge/2016/12/8/archimedes-feet

7: Pictire of Newton: https://da.wikipedia.org/wiki/Isaac_Newton

8: https://www.cctv.com/english/TouchChina/GloryofChineseCivilization/HistoricalCelebrities/Scientists/20020328/100026.html

9: Picture of Van Ceulen's Grave: http://vierappels.blogspot.com/2012/03/pi-dag.html and his book

10: https://www.maa.org/book/export/html/2287704