Buffon's Needle Problem

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Introduction

In this paper we will investigate Buffon's Needle Problem. First we will give a solution to the problem. Afterwards we will use the solution to compute the digits of π. This will be done by a stochastic simulation which will be written in the program R. Finally we will see how the estimate of π converges to the true value.

Problem Statement

Consider a floor with horizontal lines that are spaced apart by length d. You drop a needle of length l on the floor. We will consider the case the $l \leq d$. Sometimes when you drop the needle it will intersect one of the horizontal lines but other times it won't. The problem known as Buffon's Needle Problem is:

What is the probability that a randomly dropped needle will cross one of the horizontal lines.

The problem was first posed by the French mathematician *Georges Louis Leclerc Buffon* (1707 - 1788) in 1733 and reproduced with a solution by him in 1777. [1]

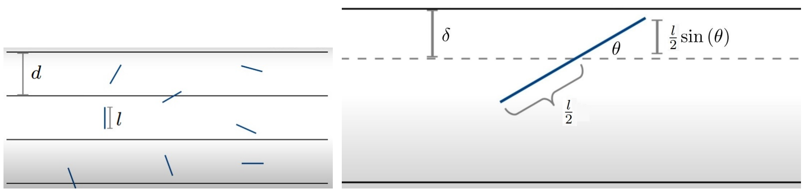
Picture of Georges Louis Leclerc Buffon. [2]

Theorem. *Buffon's Needle Problem.* The probability $P(l,d)$ that a needle of length l will land on a line, given a floor with equally spaced parallel lines a distance d apart, is

$$P(l,d) = \frac{2l}{d\pi} \ ,$$

where $l \leq d$.

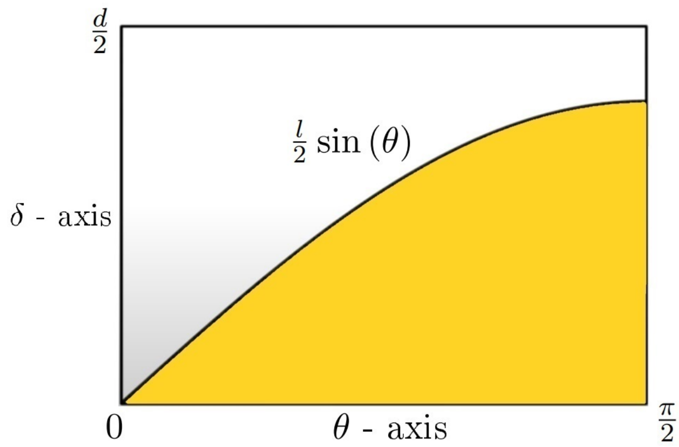
Proof. Consider all the different ways that this needle of length $l \leq d$ could fall on the floor. It can fall in different vertical positions but it could also fall at different angles. In this method we are going to consider all the different vertical positions and angles θ and compute the proportion at which the needle intersects the horizontal line. We will magnify our diagram and we're going to focus on the middle point of the needle as a reference. This is at $l/2$ distance from the of the ends.



Let's define δ and θ.

- Define δ as the distance of the middle of the needle to the closest horizontal line. Then $\delta \in \left[0, \frac{d}{2}\right]$ since the middle part of the needle can at most be halfway from one of the horizontal lines.
- Define θ as the acute angle of the needle relative to the horizontial lines. Note that $\theta \in \left[0, \frac{\pi}{2}\right]$.
- **Then the needle crosses a line if and only if $\delta \leq \frac{l}{2}\sin(\theta)$.**

Now lets consider the space of δ and θ and make a graph. The condition that the needle crosses is the yellow region in this graph. We can now compute the probability the needle crosses a horizontal line as equaling the area under the curve divided by the area of the entire sample space which is the rectangle.



Let $\mathcal{A}_1$ be the yellow area under the curve and let $\mathcal{A}_2$ be the area of the rectangle. Then

$$P(l,d) = \frac{\mathcal{A}_1}{\mathcal{A}_2}.$$

We have that $\mathcal{A}_2 = \frac{\pi}{2} \cdot \frac{d}{2}$. Now lets calculate $\mathcal{A}_1$ by calculating it as the definite integral between $\theta = 0$ and $\theta = \frac{\pi}{2}$:

$$\begin{aligned}\mathcal{A}_1 &= \int_0^{\frac{\pi}{2}} \frac{l}{2}\sin(\theta)\, d\theta \\ &= \frac{l}{2}\left[-\cos(\theta)\right]_0^{\frac{\pi}{2}} \\ &= \frac{l}{2}.\end{aligned}$$

Hence we have that

$$\begin{aligned}P(l,d) &= \frac{\frac{l}{2}}{\frac{d}{2}\frac{\pi}{2}} \\ &= \frac{l}{2}\left(\frac{2}{d}\cdot\frac{2}{\pi}\right) \\ &= \frac{2l}{d\pi}.\end{aligned}$$

□

Example

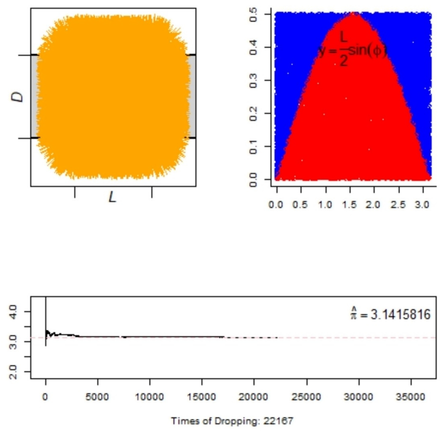
We see that the value of P is maximized when $l = d$, and in this case we get $P(l,d) = \frac{2}{\pi} \approx 64\%$.

Stochastic simulation study

We can use our theorem to numerical compute π by droping a needle on the floor with horizontial lines and repeat the proces n times. We can then count the number of drops and the number of intersections. Let l and d be equal to 1. We can then compute π, since

$$P(1,1) = \frac{2}{\pi} \Rightarrow \pi = \frac{2}{P(1,1)} \approx \frac{2}{\frac{\texttt{Number of intersections}}{\texttt{Number of drops}}}.$$

I have written a code in R that simulates this process. After 36000 simulations we get the output as shown. A video of the simulation can be found at `https://worldpifederation.org/buffon.html`.



The R Code

```
library(animation)
oopts = if (.Platform$OS.type == "windows") {
ani.options(ffmpeg =
   "C:\\Users\\Mark\\Desktop\\script\\OOP_data\\
ffmpeg-master-latest-win64-gpl-shared\\
ffmpeg-master-latest-win64-gpl-shared\\bin\\ffmpeg.exe")}
# Usually Linux users do not need to worry about
# the ffmpeg path as long as FFmpeg
# or avconv has been installed

# Let's set the number of simulations to 36000.
# It takes several seconds if 'redraw = TRUE'
ani.options(nmax = 36000, 1)
par(mar = c(3, 2.5, 0.5, 0.2), pch = 20, mgp = c(1.5,
   0.5, 0))
#buffon.needle()
```

```
bb=buffon.needle(
l = 0.99,
d = 1.0,
redraw = TRUE,
mat = matrix(c(1, 3, 2, 3), 2),
heights = c(3, 2),
col = c("lightgray", "orange", "gray", "red", "blue",
   "black", "pink"),
expand = 0.4,
type = "l",
)

# Save a video of the function with 36000 simulations
saveVideo({

        ani.options(interval = 0.05, nmax = 36000)

        bb=buffon.needle(
        l = 0.99,
        d = 1.0,
        redraw = TRUE,
        mat = matrix(c(1, 3, 2, 3), 2),
        heights = c(3, 2),
        col = c("lightgray", "orange", "gray", "red",
           "blue", "black", "pink"),
        expand = 0.4,
        type = "l",
        )


}, video.name = "buffon.mp4"
, other.opts = "-pix_fmt yuv420p -b 300k"

)
# Higher bit-rate, better quality

ani.options(oopts)
```

Variance Analysis

We can calculate the expected error in approximation π when dropping a needle n times on a plane with parallel lines. Suppose we obtain X crossings after n drops. Then the probability that a dropped needle crosses a line is $\frac{2l}{d\pi}$, hence X has a binomial(n, p) distribution with $p = \frac{2l}{d\pi}$. Let $d = 1$ and $l \in [0, 1]$. Then $E(X) = np = n\frac{2l}{\pi} \Rightarrow \frac{E(X)}{2nl} = \frac{1}{\pi}$. Lets define $Y := \frac{X}{2nl}$. We will now see how the value of l effects the variance of Y approximate $\frac{1}{\pi}$.

$$Var\,(Y) = Var\left(\frac{X}{2nl}\right) = \frac{1}{4n^2l^2}Var\,(X) = \frac{1}{4n^2l^2}n\frac{2l}{\pi}\left(1 - \frac{2l}{\pi}\right)$$
$$= \frac{1}{2\pi nl}\left(1 - \frac{2l}{\pi}\right) = \frac{1}{2\pi nl} - \frac{2l}{2\pi^2 ln} = \frac{\pi - 2l}{2ln\pi^2}.$$

The variance is a decreasing function of the needle length l. The variance have a minimum value when $l = 1$. The lower the variance the better the estimate. Then for simplicity we will assume that $d = l$. Before we will calculate the expected error we will state the Central Limit Theorem and a lemma which will be used.

Central Limit Theorem

In probability theory, the Central Limit Theorem establishes that, in almost all situations, for independent and identically distributed random variables, the sampling distribution of the standardized sample mean tends towards the standard normal distribution even if the original variables themselves are not normally distributed. [3]

Lemma. *$E|Z| = \sqrt{\frac{2}{\pi}}$ where $Z \sim N(0, 1)$.*

Proof. Use substitution: $u := -z^2/2 \Rightarrow du = -zdz \Rightarrow -\frac{1}{z}du = dz$. Then

$$\begin{aligned}
E|Z| &= \int_{-\infty}^{\infty} |z|\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= \int_{-\infty}^{0} z\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz + \int_{0}^{\infty} z\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= \int_{z=-\infty}^{z=0} \frac{-1}{\sqrt{2\pi}}e^{u}du + \int_{z=0}^{z=\infty} \frac{-1}{\sqrt{2\pi}}e^{u}du \\
&= \left|\frac{-1}{\sqrt{2\pi}}e^{-2z^2}\right|_{z=-\infty}^{z=0} + \left|\frac{-1}{\sqrt{2\pi}}e^{-2z^2}\right|_{z=0}^{z=\infty} \\
&= \sqrt{\frac{1}{2\pi}} + \sqrt{\frac{1}{2\pi}} \\
&= \sqrt{\frac{2}{\pi}}.
\end{aligned}$$

$\square$

Proposition. *The expected error for* $\frac{1}{Y}$ *is close to* $\frac{2}{\sqrt{n}}$ *with sample size* n.

Proof. From the Central Limit Theorem we have a normal approximation to the binomial variable, so

$$Y = \frac{X}{2n}$$

has approximately normal distribution with mean

$$E(Y) = \frac{np}{2n} = \frac{1}{\pi}$$

and variance

$$Var(Y) = \frac{p(1-p)n}{4n^2} = \frac{\frac{2}{\pi}\left(1-\frac{2}{\pi}\right)}{4n}.$$

Notice that $\frac{1}{Y}$ approximates π. To estimate the error in this approximation we compute

$$\frac{1}{Y} - \pi = \frac{\pi}{Y}\left(\frac{1}{\pi} - Y\right) \approx \pi^2\left(\frac{1}{\pi} - Y\right).$$

The expected error in estimating π with $\frac{1}{Y}$ is therefore

$$E\left|\frac{1}{Y} - \pi\right| \approx \pi^2 E\left|Y - \frac{1}{\pi}\right|.$$

Notice that $Y \sim N\left(\mu, \sigma^2\right)$, where $\mu = \frac{1}{\pi}$ and $\sigma^2 = \frac{\frac{2}{\pi}\left(1-\frac{2}{\pi}\right)}{4n}$. Let $Z := \frac{Y-\mu}{\sigma} \sim N(0,1)$ be the standard normal variable. Then $Y - \mu = \sigma \cdot Z$ and

$$E\left|Y - \mu\right| = \sigma \cdot E|Z| \underset{\substack{\uparrow \\ \text{lemma}}}{=} \sigma \cdot \sqrt{\frac{2}{\pi}}.$$

Hence

$$\pi^2 E\left|Y - \frac{1}{\pi}\right| = \pi^2 \cdot \sqrt{\frac{\frac{2}{\pi}\left(1-\frac{2}{\pi}\right)}{4n}} \cdot \sqrt{\frac{2}{\pi}} = \frac{\pi\sqrt{1-\frac{2}{\pi}}}{\sqrt{n}} \approx \frac{2}{\sqrt{n}}.$$

$\square$

Observations

We can use a similar argument to calculate the number of drops needed to achieve a specified amount of error, say with probability exceeding $1-\alpha$ for the significant level $\alpha = 0.05$. We see that Buffon's needle is a very inefficient way to approximate π, since the error in approximation is proportional to $\frac{1}{\sqrt{n}}$. In general to have an expected error $2 \cdot 10^{-m/2}$ we need $n = 10^m$ drops.

- For 1 correct places we need around $n = 10^2 = 100$ drops.
- For 2 correct places we need around $n = 10^4 = 10000$ drops.
- For 3 correct places we need around $n = 10^6$ drops.
- For 4 correct places we need around $n = 10^8$ drops.

Lets say we drop a needle per second. There are 31536000 seconds per year. To compute the first 6 digits of π would take about 32000 years. To compute the first 9 digits of π would take about 32 billion years. That's more than the age of the Universe which is 13.7 billion years. This would take a lot of time and motivation.

Conclusion

In this paper we investigated Buffon's Needle Problem. The probability $P(l, d)$ that a needle of length l will land on a line, given a floor with equally spaced parallel lines a distance d apart, is $P(l, d) = \frac{2l}{d\pi}$ where $l \leq d$.The probability is maximized when $l = d$ and in this case we get $P(l, d) = \frac{2}{\pi} \approx 64\%$. A program in R was coded as well which shows a stochastic simulation of the approximation of π. Two videoes has been conducted and are available on the website. The variance of the estimator of π is minimizes when $l = d$. The variance of the estimator is close to $\frac{1}{\sqrt{n}}$ for n drops which shows us that it's not a very effective approach to compute π. As matter of fact if we drop a needle every second to compute the first 9 digits of π, it would take about 32 billion years. That's more than the age of the Universe which is 13.7 billion years.

References

[1]: Buffon 1777, page 100-104.
[2]: Picture of Buffon: `https://denstoredanske.lex.dk/Georges_Louis_Leclerc_Buffon`
[3]: "Steven K. Thompson (2002) Sampling Second Edition", page 76.
[4] : "K.V Mardina, J.T. Kent and J.M. Bibby (2003) Multivariate Analysis", page 242-243.